

THE JENKINS-SERRIN PROBLEM FOR CONSTANT MEAN CURVATURE GRAPHS IN THE HEISENBERG SPACE $\text{Nil}_3(\tau)$

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ABSTRACT. In this paper we find functions over bounded domains in the 2-dimensional Euclidean space, whose graphs (in the Heisenberg space) has constant mean curvature different from zero and taking on (possibly) infinite boundary values over the boundary of the domain.

1. INTRODUCTION

The classical Dirichlet problem for the constant mean curvature and minimal surfaces (H-surfaces and minimal surfaces from now on) in the 3-dimensional Euclidean space \mathbb{R}^3 , consists in the determination of a function $u = u(x, y)$ satisfying the partial differential equation

$$(1.1) \quad (1 + q^2)r - 2pqs + (1 + p^2)t = 2H(1 + p^2 + q^2)^{3/2}$$

in a fixed domain Ω in the (x, y) -plane, and taking on assigned continuous value on the boundary of Ω .

In order to introduce the Jenkins-Serrin theory, we divide the equation (1.1) in two cases.

- For the case of minimal surfaces in \mathbb{R}^3 , the Dirichlet problem was solved by Radô in 1930, for convex domains, see [16]. Radô based his proof on the existence theorem for the parametric problem of least area. In [10] Jenkins and Serrin gave an alternate proof entirely avoiding reference to the parametric problem.

On the other hand, it was well known that in 1934 H. F. Scherk discovered his famous minimal surface which is a graph of a function defined over a square and taking on infinite boundary data. More precisely, Scherk find out the surface given by

$$z = \log \cos x - \log \cos y, \quad |x| < \pi/2, \quad |y| < \pi/2.$$

The graph of this function is a minimal surface in \mathbb{R}^3 , the function takes on plus infinite and minus infinite boundary data on alternates sides of the boundary of the square. We call this surface, the Scherk example. This Scherk example can be seen as a solution of the Dirichlet problem for the minimal curvature equation, taking on infinite boundary values over the boundary of the square.

In [10] Jenkins and Serrin developed an existence and uniqueness theory applicable to situation in which continuity of the data is set aside and which infinite boundary values are allowed on entire arcs of the boundary. Fundamental for they work was the existence of a solution of the Dirichlet problem in a convex domain with assigned boundary data, notice that this work generalize the Scherk example. From now on, we call this kind of surfaces, Jenkins-Serrin surfaces and the theory development, the Jenkins-Serrin theory.

The Jenkins-Serrin theory was extend to $\mathbb{H} \times \mathbb{R}$ (here \mathbb{H} label the 2-dimensional hyperbolic space) by Nelli and Rosenberg in [13], and to $M^2 \times \mathbb{R}$, where M^2 is an arbitrary Riemannian surface, by Pinheiro in [15]. And for non-compact domains, the Jenkins-Serrin theory was treated by Mazet L., Rodríguez, M. and Rosenberg, H. in [12].

Different of the 3-dimensional product spaces $\mathbb{H} \times \mathbb{R}$ and $\mathbb{S}^2 \times \mathbb{R}$ (here \mathbb{S}^2 label the 2-dimensional Euclidean sphere), which are homogeneous and simply connected, having 4-dimensional isometry groups, we have the Heisenberg space $\text{Nil}_3(\tau)$ and the universal cover of the space $PSL_2(\mathbb{R})$ (here $PSL_2(\mathbb{R})$ denote the preserving-orientation isometries of the hyperbolic space \mathbb{H}) which we denote by $\widetilde{PSL}_2(\mathbb{R}, \tau)$. For more details, see for instance [19], [4].

The Jenkins-Serrin theory for compact domains in $\text{Nil}_3(\tau)$ was treated by Ana Lucia Pinheiro in [14] and for the $\widetilde{PSL}_2(\mathbb{R}, \tau)$ (in the $\tau = -1/2$ case) space by Rami-Younes in [20].

- On the other hand, for the case $H \neq 0$, the Jenkins-Serrin theory for bounded domains was extended to \mathbb{R}^3 by Spruck in [18], for $\mathbb{H} \times \mathbb{R}$ and $\mathbb{S}^2 \times \mathbb{R}$ by Hauswirth, Rosenberg and Spruck in [9].

For unbounded domains, the Jenkins-Serrin theory was extended to $\mathbb{H} \times \mathbb{R}$ by Folha and Mello in [6], and in the case $M^2 \times \mathbb{R}$ (here M^2 label a Hadamard surface) by Folha and Rosenberg in [8].

If we consider complete, simply connected, 3-dimensional homogeneous manifolds having 4-dimensional isometry groups, it result open the case $H \neq 0$ for bounded domains in the Heisenberg space $\text{Nil}_3(\tau)$ and the $\widetilde{PSL}_2(\mathbb{R}, \tau)$ space. As we have remarked, the fundamental fact in the Jenkins-Serrin theory, was the existence of a solution of the Dirichlet problem with prescribed continuous boundary data, on a convex domain. Since that in [2] Dajczer and Lira solve this Dirichlet problem for general 3-manifolds carrying on a non-singular Killing field, we use the existence of such solution to extend the Jenkins-Serrin Theory to $\text{Nil}_3(\tau)$ in the case $H \neq 0$ for bounded domains. The key idea is the study the flux of monotone increasing and decreasing sequences of solutions of the Dirichlet problem along arcs where they diverge, we will use the Killing submersion in order to study the geometric behaviour of such sequences of graphs. The Jenkins-Serrin problem for unbounded domain in the $\widetilde{PSL}_2(\mathbb{R}, \tau)$ space is treated in [7] and for a Killing Submersion, see [1].

The paper is organized as follow, in Section 2 we give the details of the Heisenberg space $\text{Nil}_3(\tau)$ seen as a Riemannian submersion over the 2-dimensional Euclidean space \mathbb{R}^2 . We prove a Maximum principle and we cite the existence theorem of the Dirichlet problem. In Section 3, we deal with H -sections, and establishes the Jenkins-Serrin problem as well as we study the properties of the flux of the sequences of solutions. Finally in Section 4, we prove the main theorems.

2. PRELIMINARES

We denote by $\text{Nil}_3(\tau)$ the 3-dimensional Lie group endowed with a left invariant metric g . For each $\tau \neq 0$, $\text{Nil}_3(\tau)$ is a homogeneous simply connected Riemannian manifold.

In Euclidean coordinates $\text{Nil}_3(\tau) = (\mathbb{R}^3, g)$, where \mathbb{R}^3 label the 3-dimensional Euclidean space and $\text{Nil}_3(\tau)$ is endowed with the metric

$$(2.1) \quad g = dx^2 + dy^2 + (\tau(ydx - xdy) + dz)^2$$

The Lie group $\text{Nil}_3(\tau)$ is one of the eight Thurston's geometries (see [19]), and it is well known that there exists a Killing submersion (see [17])

$$\pi : \text{Nil}_3(\tau) \longrightarrow \mathbb{R}^2$$

$$(2.2) \quad (x, y, z) \longmapsto (x, y)$$

from $\text{Nil}_3(\tau)$ into the Euclidean 2-dimensional space \mathbb{R}^2 . That is π is a Riemannian submersion, the bundle curvature is τ and the unit vector field along the fibers is a Killing vector field. Therefore translations along the fibers are isometries. We denote this Killing vector field by ξ .

We call a vector field $Z \in \chi(\text{Nil}_3(\tau))$ vertical if it is a non-zero multiple of ξ and horizontal if $g(Z, \xi) = 0$.

In order to obtain an orthonormal frame on $\text{Nil}_3(\tau)$, we consider the canonical frame $\{e_1 = \partial_x, e_2 = \partial_y\}$ of \mathbb{R}^2 and consider the horizontal lift E_1 and E_2 of e_1 and e_2 respectively. Then the canonical orthonormal frame $\{E_1, E_2, \xi\}$ of $\text{Nil}_3(\tau)$ is given by

$$E_1 = \partial_x - \tau y \partial_z, \quad E_2 = \partial_y + \tau x \partial_z, \quad \xi = \partial_z.$$

Denote by $\bar{\nabla}$ the Riemannian connection of $\text{Nil}_3(\tau)$, so

$$\begin{aligned} \bar{\nabla}_{E_1} E_1 &= 0 & \bar{\nabla}_{E_1} E_2 &= \tau E_3 & \bar{\nabla}_{E_1} E_3 &= -\tau E_2 \\ \bar{\nabla}_{E_2} E_1 &= -\tau E_3 & \bar{\nabla}_{E_2} E_2 &= 0 & \bar{\nabla}_{E_2} E_3 &= \tau E_1 \\ \bar{\nabla}_{E_3} E_1 &= -\tau E_2 & \bar{\nabla}_{E_3} E_2 &= \tau E_1 & \bar{\nabla}_{E_3} E_3 &= 0 \end{aligned}$$

$$[E_1, E_2] = 2\tau E_3, \quad [E_2, E_3] = 0 = [E_1, E_3]$$

for more details see [3], [5], [4].

The isometry group of $\text{Nil}_3(\tau)$ has dimension 4, the isometries of $\text{Nil}_3(\tau)$ are the translations generated by the Killing vector fields

$$F_1 = \partial_x + \tau y \partial_z, \quad F_2 = \partial_y - \tau x \partial_z, \quad F_3 = \partial_z$$

and the rotations about the z -axis corresponding to

$$F_4 = -y \partial_x + x \partial_y$$

The translations corresponding to F_1 and F_2 are respectively

$$(x, y, z) \mapsto (x + t, y, z + \tau t y)$$

and

$$(x, y, z) \mapsto (x, y + t, z - \tau t x)$$

where $t \in \mathbb{R}$.

2.1. The mean curvature equation and the maximum principle. We identify the space \mathbb{R}^2 with its lift $\mathbb{R}^2 \times \{0\} \subset \text{Nil}_3(\tau)$ and for a given domain $\Omega \subset \mathbb{R}^2$, we also denote by Ω its lift to $\mathbb{R}^2 \times \{0\}$.

To a function $u \in C^0(\overline{\Omega})$ on Ω , we define the graph of u , denoted by $\Sigma(u)$, as being the set

$$(2.3) \quad \Sigma(u) = \{(x, y, u(x, y)) \in \text{Nil}_3(\tau); (x, y) \in \Omega\}$$

Throughout this paper the surface $\Sigma(u)$ will have mean curvature $H > 0$ with respect to the upward pointing normal vector of $\Sigma(u)$.

In order to obtain the normal vector N of $\Sigma(u)$, consider the function

$$u^* : \text{Nil}_3(\tau) \longrightarrow \mathbb{R}$$

$$(x, y, z) \mapsto u^*(x, y, z) = u(x, y)$$

and set $F(x, y, z) = z - u^*(x, y, z)$. Therefore, $\Sigma(u) = F^{-1}(0)$, that is $\Sigma(u)$ is the level surface of F . It is well known that the function H satisfies

$$(2.4) \quad \text{div}_{\text{Nil}_3(\tau)} \left(\frac{\overline{\nabla}_g F}{|\overline{\nabla}_g F|} \right) = -2H$$

where $\text{div}_{\text{Nil}_3(\tau)}$ and $\overline{\nabla}_g$ denote the divergence and gradient in $\text{Nil}_3(\tau)$. Thus, the upward pointing normal N is given by

$$N = \frac{\overline{\nabla}_g F}{|\overline{\nabla}_g F|}.$$

A straightforward computation shows

$$\overline{\nabla}_g F = -(u_x + \tau y)E_1 - (u_y - \tau x)E_2 + E_3$$

hence

$$W^2 := |\overline{\nabla}_g F| = 1 + (u_x + \tau y)^2 + (u_y - \tau x)^2.$$

Using the Riemannian submersion, the function u satisfies the equation (the mean curvature equation)

$$(2.5) \quad \text{div}_{\mathbb{R}^2} \left(\frac{\alpha}{W} \partial_x + \frac{\beta}{W} \partial_y \right) = 2H$$

where

$$(2.6) \quad \alpha = \tau y + u_x, \quad \beta = -\tau x + u_y, \quad \text{and} \quad W^2 = 1 + \alpha^2 + \beta^2.$$

Thus, the surface $\Sigma(u)$ has mean curvature function H if and only if u is a solution of the following PDE

$$(2.7) \quad L_H(u) := \frac{1}{W}[(1 + \beta^2)u_{xx} + (1 + \alpha^2)u_{yy} - 2\alpha\beta u_{xy}] - 2H = 0$$

for α , β and W from (2.6).

Notice that

$$N = \frac{1}{W}(-\alpha E_1 - \beta E_2 + E_3) = N^h + N^v$$

where

$$N^h = \text{horizontal part of } N = \frac{-G(u)}{W} = \frac{\alpha E_1 + \beta E_2}{W}$$

and

$$N^v = \text{vertical part of } N = \frac{E_3}{W}.$$

Consider two functions

$$u, v : \Omega \longrightarrow \mathbb{R},$$

the upwards pointing normal N_1 and N_2 of $\Sigma(u_1)$ and $\Sigma(u_2)$ are respectively

$$(2.8) \quad \begin{cases} N_1 = -\frac{G(u_1)}{W_1} + \frac{1}{W_1}E_3 \\ N_2 = -\frac{G(u_2)}{W_2} + \frac{1}{W_2}E_3 \end{cases}$$

where $W_i = W(u_i)$, $i = 1, 2$.

Notice that $\langle W_i N_i, E_3 \rangle = 1$ and

$$(2.9) \quad \begin{aligned} \langle G(u_1) - G(u_2), \frac{G(u_1)}{W_1} - \frac{G(u_2)}{W_2} \rangle &= (W_1 + W_2)(1 - \langle N_1, N_2 \rangle) \\ &= \frac{W_1 + W_2}{2} |N_1 - N_2|^2 \geq 0 \end{aligned}$$

On the other hand

$$G(u_i) = -(u_{ix} + \tau y)E_1 - (u_{iy} - \tau x)E_2$$

Setting

$$\tilde{X}_i = -\pi_*(G(u_i)) = \alpha \partial_x + \beta \partial_y = (u_{ix} + \tau y)\partial_x + (u_{iy} - \tau x)\partial_y$$

and

$$X_{u_i} = \frac{\tilde{X}_i}{W_i}$$

we conclude that

$$X_{u_1} - X_{u_2} = \nabla_0 u_1 - \nabla_0 u_2,$$

where ∇_0 denotes the gradient in the Euclidean space \mathbb{R}^2 .

Using the Riemannian submersion (2.9) becomes

$$(2.10) \quad \langle \nabla_0 u_1 - \nabla_0 u_2, X_{u_1} - X_{u_2} \rangle_{\mathbb{R}^2} \geq 0$$

Thus, we have proved the next lemma.

Lemma 2.1. *Let u_1 and u_2 be functions in $C^2(\Omega)$, $\Omega \subset \mathbb{R}^2$ and set $W_i = W(u_i)$, $i = 1, 2$. Then*

$$(2.11) \quad \langle \nabla_0 u_1 - \nabla_0 u_2, X_{u_1} - X_{u_2} \rangle \geq 0$$

with equality at a point if and only if $\nabla_0 u_1 = \nabla_0 u_2$.

Following the above notation, the mean curvature function H of the surface $\Sigma(u)$ satisfy the equation

$$(2.12) \quad \operatorname{div}_{\mathbb{R}^2}(X_u) = 2H$$

where $X_u = -\pi_* \left(\frac{G(u)}{W} \right)$.

For this situation, we prove the following maximum principle.

Theorem 2.2. *(Maximum principle) Consider $\Omega \subset \mathbb{R}^2$ a bounded domain. Let $u, v \in C^2(\Omega)$ be two functions whose graphs $\Sigma(u)$ and $\Sigma(v)$ have the same prescribed mean curvature H . Let $E \subset \partial\Omega$ be a finite set of points such that $\partial\Omega - E$ consists of smooth arcs, and suppose that u and v extend continuously to each smooth arc of $\partial\Omega - E$. If $u \geq v$ on $\partial\Omega$, then $u \geq v$ on Ω .*

Proof. Consider the set $D = \{x \in \Omega, u(x) - v(x) < 0\}$. We can translate the surfaces $\Sigma(u)$ and $\Sigma(v)$ in the ∂_z direction to assume that $u < v$ on $\Omega - E$, and by contradiction, we are supposing that D is not empty.

The boundary of D consists of proper curves in Ω which goes to points of E . We can also assume that those curves are regular, ie. $\nabla_0(u - v)$ is nonzero on ∂D . Denote by \hat{D} a connected component of D and let $\hat{D}_\epsilon \subset \hat{D}$ be the domain such that $\partial\hat{D}_\epsilon$ is the set of points in $\partial\hat{D}$ whose distance from, E is greater from $\epsilon > 0$, for ϵ small enough, together with the union $\cup C_\epsilon$ of circular arcs contained in \hat{D} which are part of circles centered at points of $\partial\hat{D} \cap E$, having radius ϵ .

From the mean curvature equation (2.12), we have

$$\int_{\hat{D}_\epsilon} \operatorname{div}_{\mathbb{R}^2}(X_u - X_v) = 0$$

and by Stokes' Theorem

$$(2.13) \quad 0 = \int_{\hat{D}_\epsilon} \operatorname{div}_{\mathbb{R}^2}(X_u - X_v) = \int_{\partial\hat{D}_\epsilon} \langle X_u - X_v, \eta \rangle$$

where η is the outward unit conormal to $\partial\hat{D}_\epsilon$.

From Lemma 2.1

$$(2.14) \quad \langle X_u - X_v, \nabla_0 u - \nabla_0 v \rangle \geq 0$$

As $\nabla_0(u - v) \neq 0$ and $u - v \equiv 0$ on $\partial\hat{D}_\epsilon - \cup C_\epsilon$, and $u - v < 0$ on \hat{D}_ϵ , the vector $\nabla_0(u - v)$ is a positive multiple of η on $\partial\hat{D}_\epsilon - \cup C_\epsilon$. Therefore from (2.14)

$$\int_{\partial\hat{D}_\epsilon - \cup C_\epsilon} \langle X_u - X_v, \eta \rangle_{\mathbb{R}^2} \geq \delta > 0.$$

On the other hand, on $\cup C_\epsilon$, we have

$$\int_{\cup C_\epsilon} \langle X_u - X_v, \eta \rangle \leq \int_{\cup C_\epsilon} |\langle X_u - X_v, \eta \rangle| \leq 2 \times \text{length}(\cup C_\epsilon)$$

thence, making ϵ tend to zero, we conclude that

$$\int_\epsilon \langle X_u - X_v, \eta \rangle > 0,$$

which contradicts (2.13). Thus D is empty and $u \leq v$ on Ω .

□

2.2. The Dirichlet problem. In [2] Dajczer and Lira have proved the existence and uniqueness of Killing graphs with prescribed mean curvature in Killing submersions. In our context, the surfaces $\Sigma(u) \subset \text{Nil}_3(\tau)$ which are the graph of the function

$$u : \Omega \subset \mathbb{R}^2 \longrightarrow \mathbb{R}$$

are Killing graphs, since they are transverse to the vertical Killing field ξ . From now on, the surfaces $\Sigma(u)$ will be called Killing graphs.

For a domain $\Omega \subset \mathbb{R}^2$ with boundary $\Gamma = \partial\Omega$, we denote the correspondent Killing cylinders by $M_0 = \pi^{-1}(\overline{\Omega})$ and $K = \pi^{-1}(\Gamma)$. We denote by H_{cyl} the inward mean curvature of K and by $Ric_{\text{Nil}_3(\tau)}$ the Ricci tensor of $\text{Nil}_3(\tau)$. Then it has showed in [2, Theorem 1] the following theorem.

Theorem 2.3. *(The Dirichelt problem) Let $\Omega \subset \mathbb{R}^2$ be a domain with compact closure and $C^{2,\alpha}$ boundary. Suppose $H_{cyl} > 0$ and*

$$\inf_{\text{Nil}_3(\tau)} Ric_{\text{Nil}_3(\tau)} \geq -2 \inf_{\Gamma} H_{cyl}^2.$$

Let $H \in C^\alpha(\overline{\Omega})$ and $\phi \in C^{2,\alpha}(\Gamma)$ be given functions. If

$$\sup_{\Omega} |H| \leq \inf_{\Gamma} H_{cyl}.$$

Then, there exists a unique function $u \in C^{2,\alpha}(\overline{\Omega})$ satisfying $u|_{\Gamma} = \phi$ whose Killing graph $\Sigma(u)$ has mean curvature H .

In this paper the ambient Ricci tensor in the v -direction is defined by

$$Ric_{\text{Nil}_3(\tau)}(v) = \sum_{i=1}^2 \langle \overline{R}(w_i, v)v, w_i \rangle$$

where \overline{R} is the curvature tensor and $\{w_1, w_2, v\}$ is an orthonormal basis.

Remark 2.4. Fix a point $p_0 \in \text{Nil}_3(\tau)$ and take a unit vector $v \in T_{p_0}\text{Nil}_3(\tau)$. Let Π be the plane orthogonal to the vector v . After a rotation around the vertical fiber passing by p_0 , we can suppose that

$$\Pi = [E_1, aE_2 + bE_3], \quad a^2 + b^2 = 1.$$

That is, $\{E_1, aE_2 + bE_3\}$ is a orthonormal basis for Π and thence

$$v = -bE_2 + aE_3.$$

Using the Riemannian connection $\overline{\nabla}$, we have

$$-2\tau^2 \leq Ric_{Nil_3(\tau)}(v) \leq 2\tau^2.$$

Denote by $\Gamma \subset \mathbb{R}^2$ a piecewise C^1 smooth simple Jordan curve. Taking the parametrization

$$c : [a, b] \longrightarrow \Gamma$$

$$s \longmapsto c(s) \in \Gamma$$

consider the geodesic curvature $k(s)$ of Γ at the point $c(s) \in \Gamma$, it is well known that when the fiber of the killing submersion are geodesic, then the mean curvature H_{cyl} of the vertical cylinder $\pi^{-1}(\Gamma)$ is given by

$$H_{cyl} = \frac{k(s)}{2}.$$

From the Ricci condition in Theorem 2.3 and remark 2.4, τ must be satisfy

$$(2.15) \quad \tau^2 \leq \inf_s \left(\frac{k(s)}{2} \right)^2$$

With this in mind, the Theorem 2.3 can be write in the next form.

Theorem 2.5. *(The Dirichelt problem) Let $\Omega \subset \mathbb{R}^2$ be a domain with compact closure and $C^{2,\alpha}$ boundary. Suppose $H_{cyl} > 0$ and $H \geq |\tau|$ for a constant H . Let $\phi \in C^{2,\alpha}(\Gamma)$ be given functions. If*

$$2H \leq k(s),$$

where $k(s)$ denotes the geodesic curvature of Γ . Then, there exists a unique function $u \in C^{2,\alpha}(\overline{\Omega})$ satisfying $u|_{\Gamma} = \phi$ whose Killing graph $\Sigma(u)$ has mean curvature H .

Once there, using Theorem 2.5 we can prove an important theorem for sequences of solutions of the mean curvature equation.

Theorem 2.6. *(Compactness theorem) Let $\{u_n\}$ be a uniformly bounded sequences of solutions of the constant mean curvature equation (2.12) in a bounded domain Ω . Then there exists a subsequence which converges uniformly on compact subsets to a solution of (2.12) in Ω .*

Proof. From Theorem 2.5, we obtain an interior estimate for the first and second derivative as well as to the solutions of the equation (2.12). Therefore on compact subdomains we have the equicontinuity of the second derivatives. Consequently by Arzela-Ascoli's Theorem, we obtain a subsequence which converges uniformly on compact subsets to a solution of (2.12) in Ω . \square

3. H -SECTIONS

A section is the image of a map

$$u : \Omega \subset \mathbb{R}^2 \rightarrow \text{Nil}_3(\tau),$$

such that $\pi \circ u = \text{Id}|_{\Omega}$, where Id label the identity map restricted to the domain Ω and π is the canonical projection from $\text{Nil}_3(\tau)$ over \mathbb{R}^2 . Thus, identifying the domain Ω with its lift to $\mathbb{R}^2 \times \{0\}$, we identify the section u with the Killing graph $\Sigma(u)$ and by simplicity we denote this surface by Σ_u .

In this section we are going to consider sections $u : \Omega \subset \mathbb{R}^2 \rightarrow \text{Nil}_3(\tau)$ whose Killing graph Σ_u has constant mean curvature (CMC-graphs) H . We call such Killing graphs, H -sections or H -surfaces. For a piece of curve $\gamma \subset \mathbb{B}$, we denote by $k(\gamma)$ its geodesic curvature. It has showed in [17, Theorem 3.3] the following theorem.

Theorem 3.1. *Let $\pi : \text{Nil}_3(\tau) \rightarrow \mathbb{R}^2$ the Killing submersion and let*

$$u : \Omega \rightarrow \text{Nil}_3(\tau)$$

be an H -section over a domain $\Omega \subset \mathbb{R}^2$. Let U_0 be a neighbourhood of an arc $\gamma \subset \partial\Omega$ and $\iota : U_0 \rightarrow \text{Nil}_3(\tau)$ a section.

Assume that for any sequence (x_n) of Ω which converge to a point $x \in \gamma$, the height of $u(x_n)$ goes to $+\infty$, that is, $u(x_n) - \iota(x_n) \rightarrow +\infty$, then γ is a smooth curve with $k(\gamma) = 2H$. If $H > 0$, then γ is convex with respect to Ω if and only if, the mean curvature vector \vec{H} of Σ_u points up along Σ_u . Moreover, Σ_u converges to the vertical H -cylinder $\pi^{-1}(\gamma)$ with respect to the C^k -topology, for any $k \in \mathbb{N}$.

As a consequence of this theorem, we prove the following lemma.

Lemma 3.2. *Let u be a solution of (2.12) in a domain Ω bounded in part by an arc γ and suppose that $m \leq u \leq M$ on γ . Then, there is a constant $c = c(\Omega)$ such that for any compact C^2 sub-arc $\gamma' \subset \gamma$,*

- (i) *If $k(\gamma') \geq 2H$, with strict inequality except for isolated points, there is a neighbourhood V of γ' in $\overline{\Omega}$ such that*

$$u \geq m - c$$

in V .

- (ii) *If $k(\gamma') > -2H$, there is a neighbourhood V of γ' in $\overline{\Omega}$ such that*

$$u \leq M + c$$

in V .

Proof. First suppose that $k > 2H$ on some sub-arc $\gamma' \subset \gamma$, let p_0 be the middle point of γ' and consider the curve γ'_1 tangent to γ' at the point p_0 having $2H < k(\gamma'_1) < k$ (with respect to the interior normal to Ω), notice that γ' is outside of Ω .

Let Γ be the curve joining the endpoints of γ'_1 having curvature $k(\gamma'_1)$ with respect to the outward pointing unit normal vector to Ω . Finally consider the sub-arc $\Gamma' \subset \Gamma$ which lies in Ω . The sub-arc Γ' intersects the arc γ' in two points, these two points determine a segment of γ' , which we will call γ' again. Now consider the domain $\Delta \subset \Omega$ bounded by Γ' and γ' , where $k(\gamma') > 2H$ and $k(\Gamma') > 2H$ with respect to the interior unit normal of Δ .

Claim 3.3. *The function $u = u|_{\Gamma'}$ satisfies $|u| \leq C$, for some constant C .*

Proof of the Claim. Denote by Σ_u the graph of u over Δ . As Σ_u is transverse to the Killing field ξ , then Σ_u is stable, therefore we have curvature estimates.

Now suppose that there is a sequence of points (p_n) in Δ such that $u(p_n)$ goes to $-\infty$. Then the tangent planes $T_{u(p_n)}\Sigma_u$ becomes vertical when n goes to $+\infty$. Using the curvature estimates, we obtain a small fixed $\delta > 0$, such that Σ_u is locally the graph of a function over a ball of radius δ , centered at the origin of $T_{u(p_n)}\Sigma_u$. From Theorem 3.1, Γ' must have curvature $k(\Gamma') = -2H$ which contradicts our assumption, analogously in the case $u(p_n)$ goes to $+\infty$, we get a contradiction, proving the claim.

It was shown in [2], that the Dirichlet problem

$$\begin{cases} L(u) = 2H, & \text{in } \Delta \\ u|_{\Gamma'} = C, & u|_{\gamma'} = m \end{cases}$$

has a sub-solution, so there is a constant $c = c(\Omega)$ such that

$$u \geq m - c.$$

On the other hand, for the case (ii), suppose there exists a decreasing sequence $\{V_n\}$ of neighbourhood of γ , that is $V_n \supset V_{n+1}$, such that for each n , there exists a point $p_n \in V_n$ with $u(p_n) = u_n > n$. Notice that the mean curvature vector is pointing upwards and the graph of the function u has constant mean curvature H . Using curvature estimates, we obtain a graph Σ_{u_n} over the tangent plane $T_{u(p_n)}\Sigma(u)$, so from Theorem 3.1, Σ_{u_n} converges to the vertical cylinder $\pi^{-1}(\gamma)$, thus γ must have constant geodesic curvature $-2H$, which is a contradiction. This completes the proof of the lemma. □

3.1. The Jenkins-Serrin problem. We are going assume that $H > 0$ with respect to the upward unit normal of the Killing surface Σ_u . Then if u tends to $+\infty$ for any approach to a boundary arc γ , necessarily the curvature $k(\gamma) = 2H$ is constant, while if u tends to $-\infty$ on γ , $k(\gamma) = -2H$. Thus we must deal with non-convex domains Ω with $\partial\Omega$ piecewise C^2 and consists of three set of open arcs $\{A_i\}$, $\{B_i\}$ and $\{C_i\}$ satisfying $k(A_i) = 2H$, $k(B_i) = -2H$ and $k(C_i) \geq 2H$ respectively. The Jenkins-Serrin problem consist in to find a solution of (2.12) in Ω taking on the boundary

values $+\infty$ on A_i , $-\infty$ on the B_i and arbitrary continuous boundary data on the C_i .

In order to give a precise announcement of the Jenkins-Serrin problem, we consider the following definitions.

Definition 3.4. (Admissible domain) We say that a bounded domain Ω is admissible if it is simply connected and its $\partial\Omega$ is piecewise C^2 and consists of three set of C^2 open arcs $\{A_i\}$, $\{B_i\}$ and $\{C_i\}$ satisfying $k(A_i) = 2H$, $k(B_i) = -2H$ and $k(C_i) \geq 2H$ respectively (with respect to the interior of Ω). We suppose that no two of the arcs A_i and no two of the arcs B_i have a common endpoint.

The Jenkins-Serrin problem is defined in the next definition.

Definition 3.5. (The Jenkins-Serrin problem) Given an admissible domain Ω , the Jenkins-Serrin problem is to find a solution of (2.12) in Ω which assumes the value $+\infty$ on each A_i , $-\infty$ on each B_i and assigned continuous data on each of the open arcs C_i . Note that the continuous data are allowed to become unbounded at the endpoints.

Remark 3.6. A solution of the Jenkins-Serrin problem is called a Jenkins-Serrin solution.

Definition 3.7. Let Ω be an admissible domain. We say that \mathfrak{P} is an admissible polygon if \mathfrak{P} is a simply domain contained in $\overline{\Omega}$ with $\partial\mathfrak{P}$ piecewise smooth consisting of arcs of constant curvature $k = \pm 2H$ with vertices chosen from among the endpoints of the families $\{A_i\}$, $\{B_i\}$ and $\{C_i\}$. For an admissible \mathfrak{P} , let $\alpha(\mathfrak{P})$ and $\beta(\mathfrak{P})$ be the total length of the arcs $\partial\mathfrak{P}$ belonging to $\{A_i\}$ and $\{B_i\}$ respectively. Finally, let $l(\mathfrak{P})$ be the perimeter of \mathfrak{P} and $A(\mathfrak{P})$ be the area of \mathfrak{P} .

3.2. Flux formulas. In this section, we deal with flux formulas for H -sections which are the crucial tool to obtain the Jenkins-Serrin solution of the Jenkins-Serrin problem.

We have denoted by X_u the negative of the projection via π of the horizontal part of the normal vector N .

Let $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$ be a solution of (2.12) in a domain Ω . Then, integrating (2.12) over Ω gives

$$(3.1) \quad 2HA(\Omega) = \int_{\partial\Omega} \langle X_u, \nu \rangle ds$$

where $A(\Omega)$ denotes the area of Ω and ν is the outer conormal to $\partial\Omega$. The right hand integral in (3.1) is called *the flux* of u across Ω . Let γ be a subarc of $\partial\Omega$ (homeomorphic to $[0, 1]$). Even if u is not differentiable on γ we can define the flux of u across γ as follows.

Definition 3.8. Choose ζ to be a simple smooth curve in Ω so that $\gamma \cup \zeta$ bounds a simple connected domain Δ_ζ . We then define the flux of u across

γ to be

$$(3.2) \quad F_u(\gamma) = 2HA(\Delta_\zeta) - \int_\zeta \langle X_u, \nu \rangle ds.$$

Notice that the integral in (3.2) is well defined as an improper integral. To see that, this definition is independent of ζ , let ζ' be another choice of curve and consider the 2-chain R with oriented boundary $\zeta' - \zeta$. By the Divergence Theorem and equation (2.12) we have

$$2HA(\Delta_{\zeta'}) - 2HA(\Delta_\zeta) = \int_{\zeta'} \langle X_u, \nu \rangle ds - \int_\zeta \langle X_u, \nu \rangle ds.$$

Therefore, the definition makes sense. Thus, if $u \in C^1(\Omega \cup \gamma)$, then

$$F_u(\gamma) = \int_\gamma \langle X_u, \nu \rangle ds.$$

Thus, we obtain the following lemma.

Lemma 3.9. *Let u be a solution of (2.12) in a domain Ω and let ζ be a piecewise C^1 curve in $\overline{\Omega}$. Then*

$$2HA(\Omega) = \int_{\partial\Omega} \langle X_u, \nu \rangle ds \quad \text{and} \quad \left| \int_\zeta \langle X_u, \nu \rangle ds \right| \leq |\zeta|.$$

Now we will prove some interesting lemmas.

Lemma 3.10. *Let Ω be a domain bounded in part by a piecewise C^2 arc ζ satisfying $k(\zeta) \geq 2H$. Let u be a solution of (2.12) in Ω which is continuous in ζ . Then*

$$(3.3) \quad \left| \int_\zeta \langle X_u, \nu \rangle ds \right| < |\zeta|.$$

Proof. It suffices to prove (3.3) for a small subarc γ of ζ . To this end let $p \in \zeta$ and let $\Omega_\epsilon = \Omega \cap B_\epsilon(p)$. Then by the Theorem 2.5, there is a solution v of (2.12) in Ω_ϵ with $v = u + 1$ on γ and $v = u$ on the remainder of the boundary. Set $w = v - u$, then by Lemma 2.1

$$0 < \int_{\Omega_\epsilon} \langle \nabla_0 w, X_v - X_u \rangle dv = \int_\gamma \langle X_v - X_u, \nu \rangle ds.$$

Thence $F_u(\gamma) < F_v(\gamma) \leq |\gamma|$. □

Lemma 3.11. *Let Ω be a domain bounded in part by an arc γ and let u be a solution of (2.12) in Ω . Then,*

- (i) *if u tends to $+\infty$ on γ , we have $\int_\gamma \langle X_u, \nu \rangle ds = |\gamma|$,*
- (ii) *if u tends to $-\infty$ on γ , we have $\int_\gamma \langle X_u, \nu \rangle ds = -|\gamma|$.*

Proof. Suppose $u \rightarrow +\infty$ on γ . Notice that the upwards normal vector N on the surface Σ_u is becoming horizontal when we approach of γ . Then at points sufficiently near to γ , we have

$$\langle X_u, \nu \rangle \geq 1 - \epsilon,$$

where ν is the outer conormal to $\gamma \subset \partial\Omega$, and $\epsilon > 0$ small enough. Consequently

$$\int_{\gamma} \langle X_u, \nu \rangle ds \geq \int_{\gamma} (1 - \epsilon) ds,$$

which implies

$$\int_{\gamma} \langle X_u, \nu \rangle ds \geq |\gamma|.$$

On the other hand, if $u \rightarrow -\infty$ on γ , we have

$$(3.4) \quad \langle X_u, \nu \rangle \leq -1 + \epsilon,$$

at points sufficiently near γ , and for $\epsilon > 0$ small enough.

As $\langle X_u, \nu \rangle < 0$, Lemma (3.9) implies

$$(3.5) \quad F_u(\gamma) \geq -|\gamma|.$$

Now from (3.4)

$$\int_{\gamma} \langle X_u, \nu \rangle ds \leq \int_{\gamma} (-1 + \epsilon) ds$$

for all $\epsilon > 0$, small enough. Thence

$$(3.6) \quad F_u(\gamma) \leq -|\gamma|.$$

We conclude from (3.5) and (3.6) that $F_u(\gamma) = -|\gamma|$. \square

The following lemma is a simple extension of Lemma 3.11.

Lemma 3.12. *Let Ω be a domain bounded in part by an arc γ and let $\{u_n\}$ be a sequence of solutions of (2.12) in Ω with each u_n continuous on γ . Then*

- (i) *if the sequence tends to $+\infty$ on compact subsets of γ while remaining uniformly bounded on compact subsets of Ω , we have*

$$\lim_{n \rightarrow +\infty} \int_{\gamma} \langle X_{u_n}, \nu \rangle ds = |\gamma|$$

- (ii) *if the sequence tends to $-\infty$ on compact subsets of γ while remaining uniformly bounded on compact subsets of Ω , we have*

$$\lim_{n \rightarrow +\infty} \int_{\gamma} \langle X_{u_n}, \nu \rangle ds = -|\gamma|$$

We also need the next lemma.

Lemma 3.13. *Let Ω be a domain bounded in part by an arc γ with $k(\gamma) = 2H$ and let $\{u_n\}$ be a sequence of solutions of (2.12) in Ω with each u_n continuous on γ . Then if the sequence diverges to $-\infty$ uniformly on compact subsets of Ω while remaining uniformly bounded on compact subsets of γ , we have*

$$\lim_{n \rightarrow +\infty} \int_{\gamma} \langle X_{u_n}, \nu \rangle ds = |\gamma|.$$

Proof. Note that the sequence $\{X_{u_n} = -\pi_*(N_{u_n}^h)\}$ converges uniformly to the outer normal ν on compact subsets of Ω . This implies

$$(3.7) \quad \lim_{n \rightarrow +\infty} \langle X_{u_n}, \nu \rangle = 1$$

We obtain the lemma by integrating equation (3.7). \square

3.3. Divergence lines. Now we focus our attention in the study of convergence of H -sections $\{u_n\}$ over a domain $\Omega \subset \mathbb{R}^2$.

Remember that, we have denoted by Gu_n the gradient in $\text{Nil}_3(\tau)$ of the function u_n . We begin this section with the next lemma.

Lemma 3.14. *Let $p \in \Omega$ and suppose that $|Gu_n(p)|$ is uniformly bounded. Then, there exists a subsequence of $\{v_n = u_n - u_n(p)\}$ converging uniformly to a solution of (2.12) in a neighbourhood of p in Ω .*

Proof. Notice that the surfaces Σ_{v_n} are simply the translation of the surfaces Σ_{u_n} along the Killing field ξ . So $N_{v_n}(q) = N_{u_n}(q)$ and $Gv_n(q) = Gu_n(q)$ for all $q \in \Omega$.

As Σ_{v_n} is stable, since the sections are transverse to the Killing field ξ , curvature estimates guarantee the existence of disks $D_n^\delta(p) \subset T_p \Sigma_{v_n}$ with small positive radius δ , independent of n (it depends only on the distance of p to $\partial\Omega$), where each Σ_{v_n} is a local graph (denoted by $\Sigma(v_n, \delta)$) over $D_n^\delta(p)$, having bounded geometry.

As $|Gv_n(p)|$ is bounded and taking into account

$$N = -\frac{Gv_n}{W_n} + \xi,$$

there exists a subsequence of $N_{v_n}(p)$ which converges to a non-horizontal vector and thus, the tangent planes associated to this subsequence converge to a non-vertical plane $\Pi(p)$.

Notice that the sequence of graphs $\{\Sigma(v_n, p)\}$ have height and slope uniformly bounded, thence there is a subsequence which converges uniformly to a H -graph over the disk $D^\delta(p) \subset \Pi(p)$. Moreover, since $\Pi(p)$ is not vertical, there exists a geodesic ball $B(p, \bar{\delta}) \subset \Omega$ centered at p of radius $\bar{\delta}$ with $0 < \bar{\delta} \leq \delta$, such that, the H -graph is a H -section. We conclude that there exists a neighbourhood of p in Ω where a subsequence of the $\{v_n\}$ converges to a solution of equation (2.12). \square

The Lemma 3.14 motives the following definition.

Definition 3.15. We say that the set

$$\mathfrak{C} = \{p \in \Omega; |Gu_n(p)| \text{ is bounded}\}$$

is the convergence set of the sequence $\{u_n\}$, and $\mathfrak{D} = \Omega - \mathfrak{C}$ is the divergence set of $\{u_n\}$.

Notice that, from Lemma 3.14, the convergence set \mathfrak{C} is an open subset of Ω .

Lemma 3.16. *Let \mathfrak{C}' be a connected bounded component of \mathfrak{C} . Then for any $p \in \mathfrak{C}'$, there exists a subsequence $\{v_{n'} = u_{n'} - u_{n'}(p)\}$ which converges uniformly on compact subsets of \mathfrak{C}' to a solution of (2.12) over \mathfrak{C}' .*

Proof. Taking a countable dense set $\{p_n\}$ in \mathfrak{C}' . According to the Lemma 3.14, each point p_m has a neighbourhood where a subsequence of $\{v_n\}$ converges to an H -section. This convergence is uniform on compact sets of the neighbourhood. By a diagonal process, we constructed a subsequence of $\{v_n\}$ which converges uniformly on compact subsets of \mathfrak{C}' to an H -section, this happens for any $p \in \mathfrak{C}'$. \square

Now, we study the divergent set \mathfrak{D} , where the sequence $\{|Gu_n(p)|\}$ diverges. If $p \in \mathfrak{D}$, $|Gv_n(p)|$ is unbounded and we can consider a subsequence $\{v_{n'}\}$ of $\{v_n = u_n - u_n(p)\}$ such that $|Gv_{n'}(p)| \rightarrow +\infty$. Therefore $\{N_{v_{n'}}(p)\}$ converges to a horizontal vector $N(p)$.

We are going to consider the identification. $\iota(\Omega) \approx \Omega$

Lemma 3.17. *Let $p \in \Omega$ and $\{u_n\}$ be a sequence of solutions of (2.12) in the domain Ω .*

- (i) *If $p \in \mathfrak{C}$, there is a subsequence of $\{v_n\}$ where $v_n = u_n - u_n(p)$, converging uniformly in a neighbourhood of $p \in \Omega$.*
- (ii) *If $p \in \mathfrak{D}$, there is a compact arc $L_p(\tilde{\delta})$ of curvature $2H$, containing p such that after passing to a subsequence, $\{N_{v_n}(p)\}$ converges to a horizontal vector whose projection via π is orthogonal to $L_p(\tilde{\delta})$, having the same direction as the curvature vector to the graph of v_n at $\iota(p) \approx p$.*

Proof. The first part of the lemma was proved in Lemma 3.16. We denote by Σ_{v_n} the graph of the function v_n over Ω . Note that $N_{v_n}(q) = N_{u_n}(q)$ and the convergence and divergence set are the same for $\{u_n\}$ and $\{v_n\}$.

Since H -sections are stables, they have curvature estimates, see [17]. The curvature estimates give us an $\delta > 0$ independent of n (δ only depends on the distance from p to $\partial\Omega$). Such that a neighbourhood of $v_n(p)$ in Σ_{v_n} is a graph in geodesic coordinates, with height and slope uniformly bounded over the disk $D_n^\delta(v_n(p))$ of radius δ centered at the origin of $T_p\Sigma_{v_n}$. We call this graph $\Sigma(v_n(p), \delta)$

Suppose that $p \in \mathfrak{D}$. Since $|Gv_n(p)|$ is unbounded, there is a subsequence of $N_{v_n}(p)$ that converges to a horizontal vector $N(p)$. So, for this subsequence, the tangent planes $T_p\Sigma_{v_n}$ converges to a vertical plane $\Pi(p)$ and the graphs $\Sigma(v_n(p), \delta)$ converge to a constant mean curvature H graph $\Sigma(p, \delta)$ over a disk of radius $\delta' \leq \delta$, centered at the origin of $\Pi(p)$. By the choice of the direction of the normal vector and the choice of $H > 0$, the limit of the curvature vectors of $\Sigma(v_n(p), \delta)$ has the same direction as the normal limit.

Taking the curve $L_p \subset \Omega$ passing through p , orthogonal to $\pi_*(N(p))$ with curvature $2H$ and the curvature vector at p having the same direction as $\pi_*(N(p))$. We want to prove $\Sigma(p, \delta') \subset \pi^{-1}(L_p)$.

Since $\Sigma(p, \delta')$ is tangent to $\pi^{-1}(L_p)$ at $v_n(p)$, if $\Sigma(p, \delta')$ is in one side of $\pi^{-1}(L_p)$, by the maximum principle, we have that $\Sigma(p, \delta') \subset \pi^{-1}(L_p)$. If this is not the case, $\Sigma(p, \delta') \cap \pi^{-1}(L_p)$ is composed by $k \geq 2$ curves passing through $v_n(p)$. These curves separate $\Sigma(p, \delta')$ in $2k$ components and the adjacent components lies in alternative sides of $\pi^{-1}(L_p)$. Moreover, the curvature vector alternates from pointing down to pointing up when one goes from one component to other. This implies that the normal vector to $\Sigma(p, \delta')$ points down and up. So, for n big enough, the normal vectors to $\Sigma(v_n(p), \delta)$ would point down and up, which does not occur.

Let $L_p(\tilde{\delta}) \subset \Omega$, $\delta' \geq \tilde{\delta}$ be the curve contained in $\Sigma(p, \delta') \cap \iota(L_p)$ which contains $p \approx \iota(p)$, and has length $2\tilde{\delta}$. Since $\Sigma(p, \delta') \subset \pi^{-1}(L_p)$, we have that, for all $q \in L_p(\tilde{\delta})$ the normal vector to $\Sigma(p, \delta')$ at q is a horizontal vector normal to $L_p(\tilde{\delta})$ at q . \square

Lemma 3.18. *Let $\{u_n\}$ be a sequence of solutions of (2.12) in Ω . Given $p \in \mathfrak{D}$, there is a curve $L \subset \Omega$ of curvature $2H$ which passes through p such that, after passing to a subsequence, the sequence of normal vectors $\{N_{u_n}|_L\}$ converges to a horizontal vector, whose projection via π is normal to L , and having the same direction as the curvature vector of L . This curve L contains the compact arc $L_p(\tilde{\delta})$ given in Lemma 3.17.*

Proof. Let L be the curve of constant curvature $2H$ in Ω , which contains $L_p(\tilde{\delta})$ joining the points of $\partial\Omega$ ($L_p(\tilde{\delta})$ is given in the Lemma 3.17). Given $p, q \in \Omega$, denote by \overline{pq} the compact arc in L between p and q . We define

$$\Lambda = \{q \in L; T(q) \text{ is true}\}$$

where

- $T(q) =$ there is a subsequence of $\{u_n\}$ such that $\{N_{u_n}|_{\overline{pq}}\}$ becomes horizontal, whose projection via π becomes orthogonal to L , having the same direction as the curvature vector of L .

We want to prove that $\Lambda = L$. Since $p \in \Lambda$, Λ is not empty. We will prove that Λ is open and closed. First, we will prove that Λ is open. Let q be a point in Λ . denote by $u_{\Lambda(n)}$ the subsequence associated to Λ . Since $\Lambda \subset \mathfrak{D}$, Lemma 3.17 give us a curve $L_q(\tilde{\delta})$ through q such that, after passing to a subsequence $\{N_{u_{\Lambda(n)}}|_{L_q(\tilde{\delta})}\}$ becomes horizontal and has the same direction as the curvature vector of $L_q(\tilde{\delta})$. Note that this subsequence of $\{N_{u_{\Lambda(n)}}|_{L_q(\tilde{\delta})}\}$ converges to a horizontal vector normal to $L_q(\tilde{\delta})$ and to L simultaneously, so $L_q(\tilde{\delta}) \subset L$, thus Λ is open.

Now, we will prove that Λ is closed. We take a convergent sequence $\{q_n\}$ in Λ , $q_n \rightarrow q \in L$. We will show that $q \in \Lambda$. For each m , there is a subsequence of $\{u_{\Lambda(n)}\}$ such that $\{N_{u_{\Lambda(n)}}|_{\overline{pq_m}}\}$ becomes horizontal with the same direction as the curvature vector of L in $\overline{pq_m}$. By the diagonal process, we obtain a subsequence of $\{u_{\Lambda(n)}\}$ such that $\{N_{u_{\Lambda(n)}}|_{\overline{pq_m}}\}$ converges to a horizontal vector having the same direction as the curvature vector of L in $\overline{pq_m}$ for all m . Then by Lemma 3.17, we

can find a curve $L_{q_m}(\delta)$ having constant curvature $2H$ through q_m (for m large, δ depends only on the distance from q to $\partial\Omega$) such that $\{N_{u_{\Lambda(n)}}|_{\overline{pq_m}}\}$ converges to a horizontal vector having the same direction as the curvature vector to $L_{q_m}(\delta)$. So, $L_{q_m} \subset L$ and since $q_m \rightarrow q$, we have that, for all m large enough, $q \in L_{q_m}(\delta)$. Consequently $q \in \Lambda$. \square

An important conclusion of the Lemma 3.18 is that the divergence set is given by $\mathfrak{D} = \bigcup_{i \in I} L_i$, where L_i is a curve called a divergence curve, having curvature $2H$. More precisely, we have the following definition.

Definition 3.19. (Divergence line) Let $L \subset \Omega$ be a curve having constant curvature $k(L) = 2H$, $p \in L$ and $\{v_n = u_n - u_n(p)\}$. If there exists a subsequence of $\{N_{v_n}|_L\}$ which converges to a horizontal vector whose projection via π is orthogonal to L , then we said that L is a divergence line of $\{u_n\}$.

Lemma 3.20. Let $\{u_n\}$ be a sequence of solutions of (2.12) in Ω . Suppose that the divergence set \mathfrak{D} of $\{u_n\}$ is composed of countable number of divergence lines. Then there exists a subsequence of $\{u_n\}$, again denoted by $\{u_n\}$ such that

- the divergence set of $\{u_n\}$ is composed of a countable number of pairwise disjoint divergence lines,

Proof. Suppose that $\mathfrak{D} \neq \emptyset$ and let L_1 be a divergence line of $\{u_n\}$, Lemma 3.17 guarantees that, after passing to a subsequence, $\{N_{u_n}(q)\}$ converges to a horizontal vector whose projection via π is orthogonal to L_1 at q for all $q \in L$. The divergence set of this sequence is contained in the divergence set of the original sequence, so the divergence set associated to this subsequence has only countable number of lines. This subsequence is still denoted by $\{u_n\}$ and its divergence set by \mathfrak{D} . If there is a line $L_2 \neq L_1$ in \mathfrak{D} , we can find a subsequence such that $\{N_{u_n}(q)\}$ converges to a horizontal vector whose projection via π is orthogonal to L_2 at q for each $q \in L_2$. This implies that $L_1 \cap L_2 = \emptyset$. In fact, if this does not occur, we take a point $q \in L_1 \cap L_2$, such that the sequence $\{N_{u_n}(q)\}$ converges to a horizontal vector whose projection is orthogonal to L_1 and L_2 at q having the same direction as the curvature vector of L_1 and L_2 . Then the uniqueness of a curve through q having constant curvature $2H$ with a given tangent vector shows that $L_1 = L_2$. We continue this process to get a subsequence of $\{u_n\}$ still denoted by $\{u_n\}$, whose divergence set is composed of a countable number of pairwise disjoint divergence lines. \square

Lemma 3.21. Let Ω be a domain bounded in part by an arc C having $k(C) \geq 2H$. Let $\{u_n\}$ be an increasing or decreasing sequence of solutions of (2.12) in Ω with each u_n continuous in $\Omega \cup C$. Suppose that γ is an interior arc of Ω of curvature $2H$ forming part of the boundary \mathfrak{D} . Then γ cannot terminate at an interior point of C if $\{u_n\}$ either diverges to $\pm\infty$ or remains uniformly bounded on compact subsets of C .

Proof. Suppose that γ is an arc in $\partial\mathcal{D}$ that terminates at an interior point p of C . By considering only a small neighbourhood of p , we may assume that C is C^2 . By Theorem 3.1, the sequence $\{u_n\}$ cannot diverge to $-\infty$ on C . Moreover, if the curvature of C is not identically $2H$ and $\{u_n\}$ remains uniformly bounded on compact subsets of C , Lemma 3.2 insures that a neighbourhood of C is contained in \mathfrak{C} a contradiction. Hence assume $k(C) \equiv 2H$. Suppose $\{u_n\}$ diverges to $+\infty$ on C and there exists exactly one such γ terminates at p . Let q be a point of γ close to p and choose s on C close to p so that the geodesic segment \overline{sq} lies in \mathfrak{C} . Let \mathfrak{T} be the triangle formed by \overline{rq} and the constant curvature $2H$ arcs \overline{qp} and \overline{pr} . Then by Lemma 3.9

$$(3.8) \quad 2HA(\mathfrak{T}) = F_{u_n}(\overline{qp}) + F_{u_n}(\overline{pr}) + F_{u_n}(\overline{rq})$$

while by Lemma 3.12

$$(3.9) \quad \lim_{n \rightarrow +\infty} F_{u_n}(\overline{qp}) = |\overline{qp}|, \quad \lim_{n \rightarrow +\infty} F_{u_n}(\overline{pr}) = |\overline{pr}|.$$

From (3.8), (3.9) and Lemma 3.10 we conclude

$$(3.10) \quad \frac{2HA(\mathfrak{T})}{|\overline{rq}|} \geq \frac{|\overline{qp}| + |\overline{pr}|}{|\overline{rq}|} - 1$$

Keeping p fixed, we move q to q' and r to r' along the same arcs so that $|\overline{q'p}| = \lambda|\overline{qp}|$ and $|\overline{pr'}| = \lambda|\overline{pr}|$ and form the triangle \mathfrak{T}' by joining q' to r' by a geodesic. Then the left hand side of (3.10) tends to zero as $\lambda \rightarrow 0$ while the right hand side of (3.10) remains uniformly positive, a contradiction. The only other possibility is that two arcs γ_1 and γ_2 terminates at p . Then again we can find a triangle $\mathfrak{T} \subset \mathfrak{C}$ whose edges are two constant curvature $2H$ arcs and a geodesic segment as before (perhaps $\partial\mathfrak{T} \cap C = \{p\}$). The same arguments gives a contradiction.

In the case the sequence remains uniformly bounded on compact subsets of C and there is exactly on γ , we choose r on C so that \mathfrak{T} is contained in \mathcal{D} . By Lemma 3.2 (ii) the sequence must be divergent to $-\infty$ on \mathcal{D} . We now reach a contradiction as above by using Lemma 3.12. If there are two arcs terminates at p , then \mathcal{D} is necessarily the convex lens domain formed by γ_1 and γ_2 (that is $\gamma_2 = \gamma_1^*$, see Remark 4.3). Choose the point q on γ_1 and r on γ_2 and form \mathfrak{T} in \mathcal{D} . Then (3.8), (3.9) and (3.10) still hold and we reach a contradiction as before. \square

We ended this section with the following theorem.

Theorem 3.22. (*Monotone convergence theorem*) Let $\{u_n\}$ be a monotonically increasing or decreasing sequence of solutions of (2.12) in a fixed domain Ω . If the sequence is bounded in a single point of Ω , there is a non-empty open set $\mathfrak{C} \subset \Omega$ such that $\{u_n\}$ converges to a solution in \mathfrak{C} . The convergence is uniform on compact subsets of \mathfrak{C} and the divergence is uniform on compact subsets of $\mathcal{D} = \Omega - \mathfrak{C}$. If \mathcal{D} is non-empty, $\partial\mathcal{D}$ consists of arcs of curvature $\pm 2H$ and parts

of $\partial\Omega$. These arcs are convex to \mathfrak{C} for increasing sequences and concave to \mathfrak{C} for decreasing sequences.

4. THE MAIN THEOREMS

In this section we deal with the solution for the Jenkins-Serrin problem, see Section 3.1 for notations and definitions. More precisely, we will prove the following two theorems.

Theorem 4.1. (Main Theorem 1) Consider the Jenkins-Serrin problem in an admissible domain Ω and suppose the family $\{C_i\}$ is empty. Then, there exists a solution for the Jenkins-Serrin problem if and only if

$$(4.1) \quad \alpha(\partial\Omega) = \beta(\partial\Omega) + 2HA(\Omega)$$

and for all admissible polygons \mathfrak{P}

$$(4.2) \quad 2\alpha(\mathfrak{P}) < l(\mathfrak{P}) + 2HA(\mathfrak{P}) \quad \text{and} \quad 2\beta(\mathfrak{P}) < l(\mathfrak{P}) - 2HA(\mathfrak{P})$$

where $\alpha(\mathfrak{P})$ and $\beta(\mathfrak{P})$ are the total length of the arcs in $\partial\mathfrak{P}$ belonging to $\{A_i\}$ and $\{B_i\}$ respectively, finally $l(\mathfrak{P})$ denotes the perimeter of \mathfrak{P} and $A(\Omega)$ denotes the area of \mathfrak{P} .

Following the notations of Theorem 4.1, we have the next theorem.

Theorem 4.2. (Main Theorem 2) Consider the Jenkins-Serrin problem in an admissible domain Ω and suppose the family $\{C_i\}$ is non-empty. Then, there exists a Jenkins-Serrin solution of the Jenkins-Serrin problem if and only if

$$(4.3) \quad 2\alpha(\mathfrak{P}) < l(\mathfrak{P}) + 2HA(\mathfrak{P}) \quad \text{and} \quad 2\beta(\mathfrak{P}) < l(\mathfrak{P}) - 2HA(\mathfrak{P})$$

for all admissible polygon \mathfrak{P} .

Before we prove the theorems, we will use the flux formulas to see that each Jenkins-Serrin solution u satisfies the equations (4.1), (4.2) and (4.3) for an admissible polygon \mathfrak{P} .

Notice that an admissible polygon \mathfrak{P} can be write in the form

$$\mathfrak{P} = \{\cup_i A_i^{\mathfrak{P}}\} \cup \{\cup_j B_j^{\mathfrak{P}}\} \cup \{\cup_k \eta_k^{\mathfrak{P}}\}$$

where $A_i^{\mathfrak{P}}, B_j^{\mathfrak{P}}$ are arcs in $\mathfrak{P} \cap \partial\Omega$ and $\{\eta_k^{\mathfrak{P}}\}$ is composed by $2H$ -curves in Ω and possible arcs C_k in $\partial\Omega$.

For an admissible polygon \mathfrak{P} , the flux formulas yields

$$F_u(\partial\mathfrak{P}) = 2HA(\mathfrak{P}).$$

Where, we conclude that

$$\begin{aligned} F_u(\{\cup_i A_i^{\mathfrak{P}}\}) &= -F_u(\mathfrak{P} - \{\cup_i A_i^{\mathfrak{P}}\}) + 2HA(\mathfrak{P}) \\ F_u(\{\cup_j B_j^{\mathfrak{P}}\}) &= -F_u(\mathfrak{P} - \{\cup_j B_j^{\mathfrak{P}}\}) + 2HA(\mathfrak{P}) \end{aligned}$$

Taking the first equality for instance, we have

$$\begin{aligned}
\alpha(\mathfrak{P}) = \sum_i |A_i^{\mathfrak{P}}| &= F_u(\{\cup_i A_i^{\mathfrak{P}}\}) \\
&= -F_u(\mathfrak{P} - \{\cup_i A_i^{\mathfrak{P}}\}) + 2HA(\mathfrak{P}) \\
&\leq |F_u(\mathfrak{P} - \{\cup_i A_i^{\mathfrak{P}}\})| + 2HA(\mathfrak{P}) \\
&= |F_u(\{\cup_j B_j^{\mathfrak{P}}\} \cup \{\eta_k^{\mathfrak{P}}\})| + 2HA(\mathfrak{P}) \\
&\leq |F_u(\{\cup_j B_j^{\mathfrak{P}}\})| + |F_u(\{\eta_k^{\mathfrak{P}}\})| + 2HA(\mathfrak{P}) \\
&= \beta(\mathfrak{P}) + |F_u(\{\eta_k^{\mathfrak{P}}\})| + 2HA(\mathfrak{P}) \\
&= l(\mathfrak{P}) - \alpha(\mathfrak{P}) - |\{\eta_k^{\mathfrak{P}}\}| + |F_u(\{\eta_k^{\mathfrak{P}}\})| + 2HA(\mathfrak{P}) \\
&< l(\mathfrak{P}) - \alpha(\mathfrak{P}) + 2HA(\mathfrak{P})
\end{aligned}$$

where we used that $F_u(B_j) = -|B_j|$ and $F_u(\eta_k) < |\eta_k|$. Thus

$$2\alpha(\mathfrak{P}) < l(\mathfrak{P}) + 2HA(\mathfrak{P}).$$

Analogously, we can see

$$2\beta(\mathfrak{P}) < l(\mathfrak{P}) - 2HA(\mathfrak{P}).$$

On the other hand, when $\{\cup_k C_k\} = \emptyset$ and $\partial\mathfrak{P} = \partial\Omega$, from flux formulas we have

$$F_u(\{\cup_i A_i^{\mathfrak{P}}\} \cup \{\cup_j B_j^{\mathfrak{P}}\}) = 2HA(\Omega)$$

that is

$$F_u(\{\cup_i A_i^{\mathfrak{P}}\}) + F_u(\{\cup_j B_j^{\mathfrak{P}}\}) = 2HA(\Omega)$$

which implies

$$\alpha(\mathfrak{P}) - \beta(\mathfrak{P}) = 2HA(\Omega).$$

Remark 4.3. (Lens domain) We call a domain D , a lens domain, if D is bounded by an arc γ of curvature $2H$ and its geodesic reflection γ^* .

In order to prove these two last theorems, we are going to prove some results.

Proposition 4.4. *Consider the Jenkins-Serrin problem in an admissible domain Ω and suppose the family $\{B_j\}$ is empty, $k(C_k) > 2H$ and the assigned data f on the arcs C_k are bounded below. Then, there exists a Jenkins-Serrin solution for this Jenkins-Serrin problem if and only if*

$$(4.4) \quad 2\alpha(\mathfrak{P}) < l(\mathfrak{P}) + 2HA(\mathfrak{P})$$

for all admissible polygon \mathfrak{P} .

Proof. Let u_n be the solution of (2.12) in Ω such that

$$u_n = \begin{cases} n & \text{on } \cup A_i \\ \min(n, f) & \text{on } \cup C_i, \end{cases}$$

such solution exists and is unique by the Dirichlet Theorem (Theorem 2.5). Moreover by the Maximum principle (Theorem 2.2), the sequence $\{u_n\}$ is monotone increasing, so the Monotone Convergence Theorem (Theorem

3.22) applies. Suppose that, there is a point $p \in \Omega$ such that $u_n(p) \rightarrow +\infty$, then there is a divergence line $L(p)$ passing through p , thus the divergence set \mathfrak{D} is not empty. Supposing that the divergence set \mathfrak{D} is composed by a countable divergence lines, so from Lemma 3.20, we have that \mathfrak{D} is the union of pairwise disjoint divergence lines.

By Lemma 3.18 an interior arc of Ω which bounds \mathfrak{D} must be of curvature $2H$ and by Lemma 3.21, it can terminates only among the endpoints of the arcs A_i or C_i . Moreover by Lemma 3.2 a neighbourhood of each C_i is contained in \mathfrak{C} . Therefore, the boundary of each component of \mathfrak{D} is an admissible polygon \mathfrak{P} with vertices among those of the A_i and C_i . Also the curvature $2H$ arcs forming the boundary which are not among the A_i are concave to \mathfrak{D} by Theorem 3.1.

By Lemma 3.9 applied to each u_n in \mathfrak{P}

$$(4.5) \quad 2HA(\mathfrak{P}) = F_{u_n}(\partial\mathfrak{P} - \cup' A_i) + F_{u_n}(\cup' A_i)$$

where $\cup' A_i$ is the union of the arcs A_i which are part of \mathfrak{P} . Then by Lemma 3.12

$$(4.6) \quad \lim_{n \rightarrow +\infty} F_{u_n}(\partial\mathfrak{P} - \cup' A_i) = \alpha(\mathfrak{P}) - l(\mathfrak{P}) = -(l(\mathfrak{P}) - \alpha(\mathfrak{P})).$$

But $|F_{u_n}(\cup' A_i)| \leq \alpha(\mathfrak{P})$, hence from (4.5)

$$2HA(\mathfrak{P}) \leq (\alpha(\mathfrak{P}) - l(\mathfrak{P})) + \alpha(\mathfrak{P})$$

that is

$$2HA(\mathfrak{P}) + l(\mathfrak{P}) \leq 2\alpha(\mathfrak{P}),$$

contradicting our assumption (4.4). Thus \mathfrak{D} is empty and the sequence converges uniformly on compact subsets of Ω to a solution u . Since each u_n is uniformly bounded in a neighbourhood of each C_i by Lemma 3.2, a standard barrier argument shows that $u = f$ on $\cup C_i$.

Since the necessity of (4.4) is clear, this completes the proof. \square

Similarly, we have the next theorem.

Proposition 4.5. *Consider the Jenkins-Serrin problem in an admissible domain Ω and suppose that the family $\{A_i\}$ is empty, $k(C_i) > 2H$ and the assigned data f on the arcs C_i are bounded above. Then, there exists a Jenkins-Serrin solution if and only if*

$$(4.7) \quad 2\beta(\mathfrak{P}) < l(\mathfrak{P}) - 2HA(\mathfrak{P})$$

for all admissible polygon \mathfrak{P} .

Now we use Proposition 4.4 and Proposition 4.5 to construct some useful barriers in order to remove the assumption $k(C_i) > 2H$.

Example 4.6. Let $B = B_\delta(P)$ be a ball of small radius δ centered at P , and let Q and R be the "antipodal" points on ∂B . choose points Q_1 and Q_2 on ∂B and symmetric with respect to the geodesic through QPR . Now let B_1 be an arc of curvature $-2H$ (as seem from P) joining Q_1 and Q_2

and set $A_1 = B_i^*$ (the geodesic reflection of B_1 across its endpoints). Let R_1 and R_2 on ∂B be reflections on ∂B of Q_1 and Q_2 (with respect to the geodesic orthogonal to QPR through P) and define B_2 and A_2 . Then for δ small compared with H , the domain B^+ bounded by A_1 , A_2 and parts of ∂B satisfies the conditions of Proposition 4.4 and similarly, the domain B^- bounded by B_1 , B_2 and parts of ∂B satisfies the condition of Proposition 4.5.

Let u^+ be the solution of (2.12) in B^+ with boundary values $+\infty$ on $A_1 \cup A_2$ and the constant value M on the remainder of the boundary. Similarly, let u^- be the solution of (2.12) in B^- with boundary values $-\infty$ on $B_1 \cup B_2$ and the constant value $-M$ on the remainder of the boundary.

With this example we obtain the following proposition.

Proposition 4.7. *Let Ω be a domain bounded in part by an arc γ and let $\{u_n\}$ be a sequence of solutions of (2.12) in Ω which converges uniformly on compact subsets of Ω to a solution u . Suppose each u_n is continuous on $\Omega \cup \gamma$, then*

- (i) *Suppose the boundary values of u_n converges uniformly on compact subsets of γ to a bounded limit f . If $k(\gamma) \geq 2H$, then u is continuous on $\Omega \cup \gamma$ and $u = f$ on γ .*
- (ii) *If $k(\gamma) = 2H$ and the boundary values of u_n diverges uniformly to $+\infty$ on compact subsets of γ , then u takes on the boundary values $+\infty$ on γ .*
- (iii) *If $k(\gamma) = -2H$ and the boundary values of u_n diverges to $-\infty$ on compact subsets of γ , then u takes on the boundary values $-\infty$ on γ .*

Proof. (i) It suffices to prove that the sequence $\{u_n\}$ is uniformly bounded in the intersection of Ω with a neighbourhood of any interior point P of γ . Orient the ball of Example 4.6, so that the geodesic joining QPR is tangent to $\partial\Omega$. We may choose the points Q_i $i = 1, 2$ and δ small enough so that the arc joining Q_2 and R_2 lies in a compact subset of Ω . Then if M is large enough

$$u_n \leq u^+ \text{ in } \Omega \cap B^+ \quad \text{and} \quad u_n \geq u^- \text{ in } \Omega \cap B^-.$$

Therefore, the sequence is uniformly bounded in a neighbourhood of P .

(ii) Let P be an interior point of γ . Similarly as in (i), we obtain that there exists M large enough so that

$$u_n \geq -M \text{ in } W = B_\epsilon(P) \cap \Omega.$$

Let v_m be the solution of (2.12) in W with boundary values m on $\gamma \cap B_\epsilon(P)$ and $-M$ on the remaining boundary. By the Maximum Principle $u_n \geq v_m$ for n sufficiently large so $u \geq v_m$ in W . In particular, $u(P) > m$ for every m and u must take on the value $+\infty$ at P .

(iii) Again for P interior to γ , $u_n \leq M$ in $W = B_\epsilon(P) \cap \Omega$. Let v_m be the solution of (2.12) in W with boundary values $-m$ on $\gamma \cap B_\epsilon(P)$ and M on the remaining boundary. By the Maximum Principle $u_n \leq v_m$ for n sufficiently large, so $u \leq v_m$ in W . Since the v_m are monotonically

decreasing (and converges to a solution with infinite boundary values on $\gamma \cap B_\epsilon(P)$), u must take on the value $-\infty$ at P . \square

We can now extend Proposition 4.4 and Proposition 4.5 to allow the arcs C_i to satisfy $k(C_i) \geq 2H$. The only change needed in the proof of Proposition 4.4 is to use part (i) of Proposition 4.7 to show that the solution takes on the required boundary data on the arcs C_i . The extension of Proposition 4.5 is more delicate. Since if $k(C_i) = 2H$ for some i , we do not know that the sequence is bounded below in a neighbourhood of C_i . However by Lemma 3.21 a neighbourhood of C_i is either contained in \mathfrak{C} or in \mathfrak{D} . We have already handled the former case. In the latter case, consider a component of \mathfrak{D} whose boundary is an admissible polygon \mathfrak{P} (with vertices among those of the B_i and C_i) containing a subset $\cup' C_i$ of the arcs C_i of curvature $2H$ and a subset $\cup' B_i$ of the arc B_i . The interior arcs of Ω which are in \mathfrak{P} are convex to \mathfrak{D} . By Lemma 3.9 applied to each u_n in \mathfrak{P}

$$(4.8) \quad 2HA(\mathfrak{P}) = F_{u_n}(\cup' B_i) + F_{u_n}(\cup' C_i) + F_{u_n}(\partial\mathfrak{P} - \cup' B_i - \cup' C_i)$$

Then by Lemma 3.12

$$(4.9) \quad \lim_{n \rightarrow +\infty} F_{u_n}(\partial\mathfrak{P} - \cup' B_i - \cup' C_i) = l(\mathfrak{P}) - \beta(\mathfrak{P}) - \Sigma' |C_i|$$

and by Lemma 3.13

$$(4.10) \quad \lim_{n \rightarrow +\infty} F_{u_n}(\cup' C_i) = \Sigma' |C_i|$$

But $|F_{u_n}(\cup' B_i)| \leq \beta(\mathfrak{P})$, hence

$$2HA(\mathfrak{P}) \geq -\beta(\mathfrak{P}) + \Sigma' |C_i| + (l(\mathfrak{P}) - \Sigma' |C_i| - \beta(\mathfrak{P})) = l(\mathfrak{P}) - 2\beta(\mathfrak{P})$$

contradicting our assumption (4.7).

Thus \mathfrak{D} is empty and the sequence converges uniformly on compact subsets of Ω to a solution u . Finally we use parts (i) and (ii) of Proposition 4.7 to show that our solution achieves the boundary values.

We state these results as

Theorem 4.8. *Consider the Jenkins-Serrin problem in an admissible domain Ω and suppose the family $\{B_i\}$ is empty and the assigned data f on the arcs C_i are bounded below. Then, there exists a solution if and only if*

$$(4.11) \quad 2\alpha(\mathfrak{P}) < l(\mathfrak{P}) + 2HA(\mathfrak{P})$$

for all admissible polygons \mathfrak{P} .

Theorem 4.9. *Consider the Jenkins-Serrin problem in an admissible domain Ω and suppose the family $\{A_i\}$ is empty and the assigned data f on the arcs C_i are bounded above. Then, there exists a solution if and only if*

$$(4.12) \quad 2\beta(\mathfrak{P}) < l(\mathfrak{P}) - 2HA(\mathfrak{P})$$

for all admissible domain.

Now we prove the Main Theorem 2 (Theorem 4.2). That is, we allow both families $\{A_i\}$ and $\{B_i\}$ to occur and allow the data f on the $\{C_i\}$ to be unbounded both above and below as we approach the endpoints.

Proof. (Main Theorem 2) By Theorem 4.8 the first condition of (4.3) guarantees the existence of a solution u^+ of (2.12) in Ω^* such that

$$u^+ = \begin{cases} +\infty & \text{on } \cup A_i \\ 0 & \text{on } \cup B_i^* \\ \max(f, 0) & \text{on } \cup C_i. \end{cases}$$

Similarly, by Theorem 4.9, the second condition of (4.3) guarantees the existence of a solution u^- of (2.12) in Ω^* such that

$$u^- = \begin{cases} -\infty & \text{on } \cup B_i \\ 0 & \text{on } \cup A_i \\ \min(f, 0) & \text{on } \cup C_i. \end{cases}$$

Now let u_n be the solution of (2.12) in Ω^* such that

$$u_n = \begin{cases} n & \text{on } \cup A_i \\ -n & \text{on } \cup B_i^* \\ f_n & \text{on } \cup C_i, \end{cases}$$

where f_n is the truncation of f above by n and below by $-n$.

By the Maximum Principle

$$u_n \leq u^+ \text{ in } \Omega^* \text{ and } u^- \leq u_n \text{ in } \Omega.$$

Therefore the sequence $\{u_n\}$ is uniformly bounded on compact subsets of Ω and a subsequence converges uniformly on compact subsets to a solution u in Ω . By Proposition 4.7, u takes on the boundary assigned data. The necessity of the condition (4.3) follows essentially as in the Theorem 4.8 and Theorem 4.9. □

Finally, we focus our attention in the proof of the Main Theorem 1.

Proof. (Main Theorem 1) let v_n be a solution of (2.12) in Ω^* with boundary values n on each A_i and 0 on each B_i^* . For $0 < c < n$ we define for $n \geq 1$

$$E_c = \{v_n - v_0 > 0\} \text{ and } F_c = \{v_n - v_0 < c\};$$

we suppress the dependence of these sets on n . Let E_c^i and F_c^i denote respectively the components of E_c and F_c whose closure contains respectively A_i and B_i^* . By the maximum principle $E_c = \cup E_c^i$ and $F_c = \cup F_c^i$. If c is sufficiently close to n , the sets $\{E_c^i\}$ will be distinct and disjoint (to see this, note that we can separate any two of the A_i by a curve joining two of the B_i^* on which $v_n - v_0$ is bounded away from n). Now we define $\mu(n)$ to be the infimum of the constants c such that the sets $\{E_c^i\}$ are distinct and disjoint. The sets $\{E_\mu^i\}$ will again be distinct although there must be at least one pair (i, j) , $i \neq j$ such that $\bar{E}_\mu^i \cap \bar{E}_\mu^j$ is nonempty. This implies that

given any F_μ^i there is some F_μ^j distinct from it. Now let u_i^+ , $i = 1, \dots, k$ be the solution of (2.12) in Ω^* taking on the boundary values $+\infty$ on A_i and 0 on the remaining boundary. This solution exists by Theorem 4.8 since the solvability condition (4.11) follows trivially from (4.1) and (4.2). Also let u_i^- be the solution of (2.12) in the domain $\tilde{\Omega}$ bounded by $\cup A_i, B_i^*, \cup_{j \neq i} B_j$ taking on the boundary values $-\infty$ on $\cup_{j \neq i} B_j$ and 0 on the remaining of the boundary. In order to know that the solution exists by Theorem 4.9, we need to verify (4.12), thus we need only consider the admissible polygon $\tilde{\mathfrak{P}}$ in $\tilde{\Omega}$ which contain the lens domain L formed by B_i and B_i^* . Let \mathfrak{P} be the corresponding admissible polygon for Ω formed by deleting L . By (4.1) and (4.2) we have

$$2\beta = 2(|B_i| + \sum_{i \neq j}^{\circ} |B_j|) \leq l(\mathfrak{P}) - 2HA(\mathfrak{P})$$

or equivalently

$$2\bar{\beta} = 2 \sum_{i \neq j}^{\circ} |B_i| \leq l(\tilde{\mathfrak{P}}) - 2HA(\tilde{\mathfrak{P}}) + (2HA(L) - 2|B_i|).$$

However since a solution (for example v_n) exists in L , we have by Lemma 3.10 $2HA(L) < 2|B_i|$ so condition (4.12) is satisfied.

We now set

$$u^+(p) = \max_i \{u_i^+(p)\} \text{ in } \Omega^* \text{ and } u^-(p) = \min_i \{u_i^-(p)\} \text{ in } \Omega.$$

We note that if we compare each u_i^+ to a fixed bounded solution in Ω^* (which exists since Ω is admissible), then by the maximum principle there is a constant $N > 0$ such that $u_i^+ > -N$, $i = 1, \dots, k$. Finally we set $u_n = v_n - \mu(n)$.

We now claim that

$$u_n \leq u^+ + M \text{ in } \Omega^* \text{ and } u_n \geq u^- - M \text{ in } \Omega.$$

where $M = N + \sup_{\Omega^*} |v_0|$. Suppose $u_n > v_0$ at some point p . Then

$$v_n - v_0 > \mu(n)$$

at p , so that p is in some E_μ^i . Applying the maximum principle in the domain E_μ^i , we obtain

$$u_n \leq u_i^+ + N + \sup_{E_\mu^i} |v_0| \leq u^+ + M \text{ at } p.$$

On the other hand, suppose $u_n < v_0$ at some point $p \in \Omega$. Then $v_n - v_0 > \mu(n)$ at p , so that p is in some F_μ^i . By what has been shown above, there is a corresponding $j = j(i)$ so that $F_\mu^i \cap F_\mu^j = \emptyset$, we obtain

$$u_n \geq u_j^- - \sup_{F_\mu^i} |v_0| \geq u^- - M.$$

Therefore the claim is justified and the sequence $\{u_n\}$ is uniformly bounded on compact subsets of Ω . By the compactness principle, a subsequence $\{u_n\}$ converges uniformly on compact subsets of Ω to a solution u . We still must show that u takes on the required boundary values. We observe that a subsequence $\mu(n)$ diverges to $+\infty$ otherwise we can extract a subsequence converging to a finite limit μ_0 . Each u_n would then be bounded below in Ω^* uniformly in n , and the boundary values of u_n would tend uniformly on compact subsets of $\cup B_i^*$ to $-\mu_0$ and diverge uniformly to $+\infty$ on $\cup A_i$. Once again we could find a subsequence converging uniformly on compact subsets to a solution in Ω^* . By Proposition 4.7 we would have

$$v = \begin{cases} +\infty & \text{on } \cup A_i \\ -\mu_0 & \text{on } \cup B_i^* \end{cases}$$

We can now obtain a contradiction to (4.1) by a flux argument. By Lemma 3.9

$$(4.13) \quad 2HA(\Omega) = F_v(\Sigma A_i) + F_v(\Sigma B_i)$$

while by Lemma 3.10 and Lemma 3.11

$$(4.14) \quad |F_v(\Sigma B_i)| < \beta \text{ and } F_v(\Sigma A_i) = \alpha$$

Combining (4.13) and (4.14) gives $\alpha - \beta < 2HA(\Omega)$, a contradiction. In the same way, we see $n - \mu(n)$ diverges to $+\infty$. Summing up we have shown that the boundary values of u_n , namely, $-\mu(n)$ on $\cup B_i^*$ and $n - \mu(n)$ on $\cup A_i$ diverges to $-\infty$ and $+\infty$ respectively. Therefore u_n diverges to $-\infty$ on $\cup B_i$. Since the necessity conditions (4.1) and (4.2) is straightforward, we conclude the theorem. \square

We ended this paper with the following maximum principle at the infinity, the prove of this theorem is similar to this one in [14, Theorem 4.1]

Theorem 4.10. (*Maximum principle at infinity*) Let $\Omega \subset \mathbb{R}^2$ be an admissible domain, suppose that the family $\{C_k\}$ is nonempty, and let u, v be solutions of the Dirichlet problem in $\text{Nil}_3(\tau)$, with the same continuous values on each arc $C_k \subset \partial\Omega$. Then $u = v$ on Ω .

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